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SOME NEW RESULTS ON DISTANCE ENERGY OF GRAPHS

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Abstract: The sum of absolute values of eigenvalues of distance matrix of a graph is defined as distance energy of graph. We investigate distance energy for the larger graph obtained from any arbitrary graph by means of various graph operations. The concept of equienergetic graph is also explored in the context of distance energy of graph.

Keywords and Phrases: Eigenvalues, Graph Energy, Distance Matrix, Distance Energy.

2020 Mathematics Subject Classification: 05C50, 05C76.

1. Introduction

Let G be a simple connected graph on n vertices and $v_1, v_2, ..., v_n$ be the vertices of a graph G then distance between any two vertices is defined as the length of shortest path between them. The distance between vertices v_i and v_j is denoted by d_{ij}. The diameter of graph G is denoted by $diam(G)$ and is defined as the maximum distance between any pair of vertices of G [3, 5].

The concept of graph energy was introduced by Gutman [11] and it is defined as the sum of absolute values of eigenvalues of adjacency matrix of graph G.

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$$

where $\lambda_1, \lambda_2, ..., \lambda_n$ are eigenvalues of adjacency matrix $A(G)$ of graph G. For basic concept and recent research on graph energy $\mathcal{E}(G)$ we refer to [1, 6, 10, 12, 26, 27]. Two non-isomorphic graphs G_1 and G_2 of same order are said to be equienergetic if they have same energy, $\mathcal{E}(G_1) = \mathcal{E}(G_2)$ [23].

The distance matrix of G is denoted by $D(G)$ is an $n \times n$ matrix whose $(i, j)^{th}$ entry is d_{ij}, $1 \leq i, j \leq n$ [3, 4]. The eigenvalues $\mu_1, \mu_2, ..., \mu_n$ of distance matrix $D(G)$ are called distance eigenvalues or $D-$eigenvalues of graph G and the collection of $D-$eigenvalues is called the $D-$spectrum of graph G. Since the distance matrix $D(G)$ is symmetric, all its eigenvalues are real and can be ordered as $\mu_1 \geq \mu_2 \geq ... \geq \mu_n$.

The characteristic polynomial and eigenvalues of distance matrix of graph G were extensively studied in [7, 8, 9, 13, 15, 28].

In 2008, Indulal *et al.* [16] have introduced the concept of distance energy $\mathcal{E}_D(G)$ which is defined as

$$\mathcal{E}_D(G) = \sum_{i=1}^{n} |\mu_i|$$

The distance energy was conceived in full analogy to the usual graph energy $\mathcal{E}(G)$. Two non-isomorphic graphs G_1 and G_2 of same order are said to be $D-$equienergetic if they have same distance energy, $\mathcal{E}_D(G_1) = \mathcal{E}_D(G_2)$ [16]. More results on this concepts is further explored in [16, 17, 18, 24, 28]. The graphs considered here are connected and of at most diameter 2 as it has been proved by Moon and Moser [22], Klee and Larman [19] and Bollobás [2] that almost all graphs are of diameter two.

Let G be any graph with diameter 2 then $d(u, v) = 1$, if u and v are adjacent in G and $d(u, v) = 2$, if u and v are adjacent in $\overline{G}$, where $\overline{G}$ is the complement of a graph G. Thus, the distance matrix $D(G)$ of graph G can be written as $A(G) + 2A(\overline{G})$.

Definition 1.1. [14] *For the matrices $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{p \times q}$ the Kronecker product of A and B is defined as the matrix*

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

Proposition 1.2. [14] *If λ is an eigenvalue of matrix $A = [a_{ij}]_{m \times m}$ with corresponding eigenvector x, and μ is an eigenvalue of matrix $B = [b_{ij}]_{n \times n}$ with corresponding eigenvector y. Then $\lambda\mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y$.*

The present work is aimed to obtain distance energy of larger graph obtained from a given graph by means of some graph operations and some $D-$equienergetic graph have been investigated.

2. Distance Energy of $m-$Shadow Graph

Definition 2.1. [25] *The m-Shadow graph $D_m(G)$ of a connected graph G is constructed by taking m copies of G say $G_1, G_2, ..., G_m$. Then join each vertex u in G_i to the neighbors of the corresponding vertex v in G_j, $1 \leqslant i, j \leqslant m$.*

The following result relates the distance energy of graph G and its $m-$Shadow.

Theorem 2.2. *Let G be a graph of diameter at most 2 and $\mu_1, \mu_2, ..., \mu_n$ are distance eigenvalues with $|\mu_i| \geq \frac{2(m-1)}{m}$, except zero for each i, then*

$$\mathcal{E}_D(D_m(G)) = m\mathcal{E}_D(G) + 2(m-1)\theta + 2(mn-n),$$

where θ is the difference between the number of positive and negative distance eigenvalues of graph G.

Proof. Let G be a graph with $v_1, v_2, \cdots, v_n$ as vertices of then its distance matrix $D(G)$ is given by

$$D(G) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \vdots \\ \boldsymbol{v_n} \end{array} \begin{array}{c} \begin{array}{ccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \cdots & \boldsymbol{v_n} \end{array} \\ \begin{bmatrix} 0 & d_{12} & d_{13} & \cdots & d_{1n} \\ d_{21} & 0 & d_{23} & \cdots & d_{2n} \\ d_{31} & d_{32} & 0 & \cdots & d_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & d_{n3} & \cdots & 0 \end{bmatrix} \end{array}$$

Now, consider m-copies $G_1, G_2, ..., G_m$ of graph G and then join each vertex of u of graph G_i to the neighbors of the corresponding vertex v in graph G_j, $1 \leqslant i, j \leqslant m$ to obtain $m-$Shadow $D_m(G)$. Then the distance matrix of graph $D_m(G)$ can be written as

$$D(D_m(G)) = \begin{bmatrix} D(G) & D(G)+2I & \cdots & D(G)+2I \\ D(G)+2I & D(G) & \cdots & D(G)+2I \\ \vdots & \vdots & \ddots & \vdots \\ D(G)+2I & D(G)+2I & \cdots & D(G) \end{bmatrix}$$

$$D(D_m(G)) + 2I_{mn} = \begin{bmatrix} D(G)+2I & D(G)+2I & \cdots & D(G)+2I \\ D(G)+2I & D(G)+2I & \cdots & D(G)+2I \\ \vdots & \vdots & \ddots & \vdots \\ D(G)+2I & D(G)+2I & \cdots & D(G)+2I \end{bmatrix}$$

That is,

$$D(D_m(G)) + 2I_{mn} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \otimes (D(G) + 2I)$$

Therefore, $D(D_m(G)) + 2I_{mn} = J_m \otimes (D(G) + 2I)$
where J_m is a matrix of order m with all the entries are 1. Since, we know that the spectrum of J_m is $\begin{pmatrix} 0 & m \\ m-1 & 1 \end{pmatrix}$. Hence, by Proposition 1.2

$$Spec_D(D_m(G) + 2I_{mn}) = \begin{pmatrix} 0 & m(\mu_1 + 2) & m(\mu_2 + 2) & ... & m(\mu_n + 2) \\ mn - n & 1 & 1 & ... & 1 \end{pmatrix}$$

where $\mu_i, i = 1, 2, ..., n$ are eigenvalues of $D(G)$.
Therefore,

$$Spec_D(D_m(G)) = \begin{pmatrix} -2 & m\mu_1 + 2(m-1) & m\mu_2 + 2(m-1) & ... & m\mu_n + 2(m-1) \\ mn - n & 1 & 1 & ... & 1 \end{pmatrix}$$

Since, $|\mu_i| \geq \frac{2(m-1)}{m}, i = 1, 2, ..., n$

$$\begin{aligned} \left| \mu_i + \frac{2(m-1)}{m} \right| &= |\mu_i| + \frac{2(m-1)}{m}, \;\; if\ \mu_i \geq 0 \\ &= |\mu_i| - \frac{2(m-1)}{m}, \;\; if\ \mu_i < 0 \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{E}_D(D_m(G)) &= \sum_{i=1}^{n} |m\mu_i + 2(m-1)| + \sum_{i=1}^{mn-n} |-2| \\ &= m \sum_{i=1}^{n} \left| \mu_i + \frac{2(m-1)}{m} \right| + 2(mn - n) \\ &= m \left(\sum_{\mu_i \geq 0} \left(|\mu_i| + \frac{2(m-1)}{m} \right) + \sum_{\mu_i < 0} \left(|\mu_i| - \frac{2(m-1)}{m} \right) \right) + 2(mn - n) \\ &= m \left(\sum_{\mu_i \geq 0} |\mu_i| + \sum_{\mu_i < 0} |\mu_i| + \frac{2(m-1)}{m} \left(\sum_{\mu_i \geq 0} 1 - \sum_{\mu_i < 0} 1 \right) \right) + 2(mn - n) \\ &= m \left(\mathcal{E}_D(G) + \frac{2(m-1)}{m} \theta \right) + 2(mn - n) \\ &= m\mathcal{E}_D(G) + 2(m-1)\theta + 2(mn - n). \end{aligned}$$

Illustration 2.3. *Consider cycle C_4 and its 3-shadow graph $D_3(C_4)$. It is obvious that $\mathcal{E}_D(C_4) = 8$ as $Spec_D(C_4) = \begin{pmatrix} 0 & -2 & 4 \\ 1 & 2 & 1 \end{pmatrix}$ Now, the distance matrix of $D_3(C_4)$ is given as follow*

$$D(D_3(C_4)) = \begin{bmatrix} D(C_4) & D(C_4)+2I & D(C_4)+2I \\ D(C_4)+2I & D(C_4) & D(C_4)+2I \\ D(C_4)+2I & D(C_4)+2I & D(C_4) \end{bmatrix}$$

so, $Spec_D(D_3(C_4)) = \begin{pmatrix} -2 & 4 & 16 \\ 10 & 1 & 1 \end{pmatrix}$. Hence, $\mathcal{E}_D(D_3(C_4)) = 40$.

3. Distance Energy of Extended $m-$Shadow Graph

Definition 3.1. *The extended m-Shadow graph $D^*_m(G)$ of a connected graph G is constructed by taking m copies of G, say $G_1, G_2, ..., G_m$, then join each vertex u in G_i to the neighbors of the corresponding vertex v and with v in $G_j, 1 \leqslant i, j \leqslant m$.*
The following result relates the distance energy of graph G and its extended $m-$Shadow.

Theorem 3.2. *Let G be a graph of at most diameter two with distance eigenvalues $\mu_1, \mu_2, ..., \mu_n$ and $|\mu_i| \geq \frac{m-1}{m}$ except zero for each i, then*

$$\mathcal{E}_D(D^*_m(G)) = m\ \mathcal{E}_D(G) + (m-1)\theta + (mn - n),$$

where θ is the difference between the number of positive and negative distance eigenvalues of graph G.

Proof. Let G be a graph of order n then the distance matrix of G can be written as

$$D(G) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \vdots \\ \boldsymbol{v_n} \end{array} \begin{array}{c} \begin{array}{ccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \cdots & \boldsymbol{v_n} \end{array} \\ \begin{bmatrix} 0 & d_{12} & d_{13} & \cdots & d_{1n} \\ d_{21} & 0 & d_{23} & \cdots & d_{2n} \\ d_{31} & d_{32} & 0 & \cdots & d_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & d_{n3} & \cdots & 0 \end{bmatrix} \end{array}$$

Now, consider m-copies $G_1, G_2, ..., G_m$ of graph G and then join each vertex of u of graph G_i to the neighbors of the corresponding vertex v and also with v in graph $G_j,\ 1 \leqslant i, j \leqslant m$ to obtain extended $m-$Shadow $D^*_m(G)$. Then the distance matrix of graph $D^*_m(G)$ can be written as

$$D(D^*_m(G)) = \begin{bmatrix} D(G) & D(G)+I & \cdots & D(G)+I \\ D(G)+I & D(G) & \cdots & D(G)+I \\ \vdots & \vdots & \ddots & \vdots \\ D(G)+I & D(G)+I & \cdots & D(G) \end{bmatrix}$$

so,

$$\begin{aligned} D(D^*_m(G)) + I_{mn} &= \begin{bmatrix} D(G)+I & D(G)+I & \cdots & D(G)+I \\ D(G)+I & D(G)+I & \cdots & D(G)+I \\ \vdots & \vdots & \ddots & \vdots \\ D(G)+I & D(G)+I & \cdots & D(G)+I \end{bmatrix} \\ &= J_m \otimes [D(G)+I] \end{aligned}$$

Since, we know that $Spec(J_m) = \begin{pmatrix} m & 0 \\ 1 & m-1 \end{pmatrix}$. Hence, from Proposition 1.2 we have,

$$Spec\,(D(D^*_m(G)) + I_{mn}) = \begin{pmatrix} m(\mu_i + 1) & 0 \\ 1 & mn-n \end{pmatrix}.$$

Therefore,

$$Spec\,(D(D^*_m(G))) = \begin{pmatrix} m\mu_i + (m-1) & -1 \\ n & mn-n \end{pmatrix}$$

Since, $|\mu_i| \geq \frac{m-1}{m}, i = 1, 2, ..., n$

$$\begin{aligned} \left|\mu_i + \frac{m-1}{m}\right| &= |\mu_i| + \frac{m-1}{m}, \quad if\ \mu_i \geq 0 \\ &= |\mu_i| - \frac{m-1}{m}, \quad if\ \mu_i < 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{E}_D(D^*_m(G)) &= \sum_{i=1}^{n} |m\mu_i + (m-1)| + \sum_{i=1}^{mn-n} |-1| \\ &= m \sum_{i=1}^{n} \left|\mu_i + \frac{m-1}{m}\right| + (mn-n) \\ &= m\left(\sum_{\mu_i \geq 0} \left(|\mu_i| + \frac{m-1}{m}\right) + \sum_{\mu_i < 0} \left(|\mu_i| - \frac{m-1}{m}\right)\right) + (mn-n) \end{aligned}$$

$$= m\left(\sum_{\mu_i\geq 0}|\mu_i| + \sum_{\mu_i<0}|\mu_i| + \frac{m-1}{m}\left(\sum_{\mu_i\geq 0}1 - \sum_{\mu_i<0}1\right)\right) + (mn-n)$$
$$= m\left(\mathcal{E}_D(G) + \frac{m-1}{m}\theta\right) + (mn-n)$$
$$= m\,\mathcal{E}_D(G) + (m-1)\theta + (mn-n)$$

4. Distance Energy of Square of Graph

Definition 4.1. *Let G be any graph then, square of graph G is denoted by G^2 and it is a graph with vertex set $V(G)$ and two vertices are adjacent if they are having at most distance 2 in graph G.*

Theorem 4.2. *Let G be any graph with diameter at most 2 then $\mathcal{E}_D(G^2) = 2(n-1)$.*
Proof. Let G be a graph with diameter at most 2 then, G^2 becomes a complete graph. So, distance matrix of graph G^2 is same as adjacency matrix of complete graph K_n. Hence, $\mathcal{E}_D(G^2) = \mathcal{E}(K_n) = 2(n-1)$.

5. Distance Equienergetic Graphs

Two non-isomorphic graphs G_1 and G_2 of same order are said to be $D-$equienergetic if they have same distance energy, $\mathcal{E}_D(G_1) = \mathcal{E}_D(G_2)$. We have also identify some D-equienergetic graphs in the following theorems.

Theorem 5.1. *Let G_1 and G_2 are two distance equienergetic graphs then $D_m(G_1)$ and $D_m(G_2)$ are also distance equienergetic graphs if and only if they have same number of positive and negative distance eigenvalues.*
Proof. It is obvious as if G_1 and G_2 are distance equienergetic graphs with same number of positive and negative distance eigenvalues of graphs then, $D_m(G_1)$ and $D_m(G_2)$ are distance equienergetic graphs. Conversely, if $D_m(G_1)$ and $D_m(G_2)$ are distance equienergetic graphs then from Theorem 2.2, the number of positive and negative eigenvalues have to be same.

Theorem 5.2. *Let G_1 and G_2 are two distance equienergetic graphs then $D^*_m(G_1)$ and $D^*_m(G_2)$ are also distance equienergetic graphs if and only if they have same number of positive and negative distance eigenvalues.*
Proof. It is clear as if G_1 and G_2 are distance equienergetic graphs with same number of positive and negative distance eigenvalues of graphs then, $D^*_m(G_1)$ and $D^*_m(G_2)$ are distance equienergetic graphs. Conversely, if $D^*_m(G_1)$ and $D^*_m(G_2)$ are distance equienergetic graphs then from Theorem 3.2, the number of positive and negative eigenvalues have to be same.

Theorem 5.3. *Let G_1 and G_2 be any two graphs of same order and of at most* *diameter 2 then* G_1^2 *and* G_2^2 *are distance equienergetic graphs.*

Proof. Proof is obvious from Theorem 4.2.

M. Liu [21] have proved that for $n \geq 6$, $K_{2,n-2}$ and $K_{3,n-3}$ are distance non-cospectral D-equienergetic graphs where $K_{a,b}$ is complete bipartite graph on $a+b$ vertices. Also, they have proved that $K_{n,n}$ and $W_{n,n}$ are D-equienergetic graphs where $W_{n,n}$ is the graph obtained from $K_{n,n}$ by deleting n independent edges from $K_{n,n}$.

Let $CP(n)$ be the graph of order $2n$ with $2n-2$ regularity which is obtained from complete graph K_{2n} by deleting n independent edges is also known as cocktail party graph. We obtain the following result for D-equienergetic graph.

Theorem 5.4. $D_2^*(W_{n,n})$ *and* $D_2^*(CP(n))$ *are distance non co-spectral* D*-equienergetic graphs.*

Proof. Since we know that $Spec_D(CP(n)) = \begin{pmatrix} 2n & -2 & 0 \\ 1 & n & n-1 \end{pmatrix}$.

Hence, $Spec_D(D_2^*(CP(n))) = \begin{pmatrix} 4n+1 & -3 & 1 & -1 \\ 1 & n & n-1 & 2n \end{pmatrix}$ and so, $\mathcal{E}_D(D_2^*(CP(n))) = 10n.$

Also, we know that $Spec_D(W_{n,n}) = \begin{pmatrix} 3n & n-4 & -4 & 0 \\ 1 & 1 & n-1 & n-1 \end{pmatrix}$.

So, $Spec_D(D_2^*(W_{n,n})) = \begin{pmatrix} 6n+1 & 2n-7 & -7 & 1 & -1 \\ 1 & 1 & n-1 & n-1 & 2n \end{pmatrix}$.

Therefore, $\mathcal{E}_D(D_2^*(W_{n,n})) = 10n.$

Hence, $D_2^*(W_{n,n})$ and $D_2^*(CP(n))$ are distance non co-spectral D-equienergetic graphs.

6. Conclusion

The distance energy of standard graphs is already available in the literature [16]. We focus on the following problems:

Problems:

1. Is it possible to increase the distance energy of the given graph?

2. To investigate the graphs which are $D-$equienergetic

The answer is affirmative as it has been shown that the distance energy can be increased by means of various graph operations on a given graph. In addition to this some new families of distance equienergetic graphs are investigated.

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